FACTORIZATION OF POLYNOMIALS OVER BANACH ALGEBRAS(1)

BY

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Introduction. A Banach algebra in this paper will be understood to mean a commutative, semi-simple Banach algebra with multiplicative unit e. By the carrier space Φ_A of the Banach algebra A, we mean the space of multiplicative linear functionals on A to C, the complex numbers, and it is to be endowed with the usual weak* topology (cf. [6]). For $a \in A$, $\hat{a}$ denotes the Gelfand transform of a defined on Φ_A and $\hat{A}$ will denote the collection of such functions. We set the following notation. x will be used to denote an indeterminate over A as well as over $\hat{A}$ and C. If $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i$ is a polynomial over A, let $\hat{\alpha}(x)$ and $\alpha_h(x)$ denote, respectively, $\sum_{i=0} \hat{\alpha}_i x^i$ and $\sum_{i=0}^{n} \hat{\alpha}_i(h) x^i$, $h \in \Phi_A$.

The set $Z(\alpha(x), A) = \{(h,\lambda) \in \Phi_A \times C : \alpha_h(x) \neq 0$ and $\alpha_h(\lambda) = 0\}$, $\alpha(x) \in A[x]$, plays an important role in the present paper. $Z(\alpha(x), A)$ is topologized with the relative product topology from $\Phi_A \times C$. The mapping π is defined by $\pi(h,\lambda) = h, (h,\lambda) \in Z(\alpha(x), A)$. The multiplicity function M of $\alpha(x)$ is defined as follows: for $(h,\lambda) \in Z(\alpha(x), A), M(h,\lambda)$ is equal to the multiplicity of λ as a root of $\alpha_h(x) = 0$.

In §1 we introduce the concept of M-neighborhood of a point in $Z(\alpha(x), A)$. We say that $W \subset Z(\alpha(x), A)$ is a M-neighborhood of $(h_0, \lambda_0) \in Z(\alpha(x), A)$ if W is a neighborhood in $Z(\alpha(x), A)$ of (h_0, λ_0) and if, for each $h \in \pi(W)$, $M(h_0, \lambda_0)$ is equal to the sum of the values of M at the points (h,λ) in $\pi^{-1}(h) \cap W$. Proposition 1.1 states that M-neighborhoods exist and that they form a base for the neighborhood system at each point of $Z(\alpha(x), A)$. The remainder of this section contains most of the topological lemmas needed for our work on factorization. Particular attention is paid to the case where $Z(\alpha(x), A)$ contains a compact open subset K ($\pi(K) = \Phi_A$) on which M is constant. When this condition obtains, K and Φ_A decompose topologically and this decomposition in turn forces $\alpha(x)$ to factor.

The main factorization theorem (2.1) says that if $\alpha(x) \in A[x]$ and if K ($\pi(K) = \Phi_A$) is a compact open subset of $Z(\alpha(x), A)$, then there exists a monic polynomial $\beta(x) \in A[x]$ such that $\beta(x)$ is a factor of $\alpha(x)$, $Z(\beta(x), A) = K$ and $Z(\alpha(x)/\beta(x), A) = Z(\alpha(x), A) \sim K$. A more detailed description of the factorization of monic polynomials follows. For example, it is shown that if A is indecomposable

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and if M is locally constant on $Z(\alpha(x), A)$, then the factorization of $\alpha(x)$ into irreducible factors is unique (to within units).

§3 deals with the problem of finding zeros of a given polynomial over A in A and in extensions of A which are also Banach algebras. Theorem 3.2 extends, for semi-simple Banach algebras, a theorem (4.3 [1]) of Arens and Calderón. Our theorem states that if $\alpha(x) \in A[x], f \in C(\Phi_A)$ such that $\hat{\alpha}(f) = 0$ and $M(\cdot, f(\cdot))$ is locally constant on Φ_A, then $f \in \hat{A}$.

1. **Some topological results.** Let A be a Banach algebra and

$$\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i \in A[x].$$

(The results of this section do not depend on the semi-simplicity of A.) The topology in Φ_A is to be the usual weak* topology. Then the topology in $Z(\alpha(x), A)$ induced by the product topology in $\Phi_A \times C$ is precisely the weakest topology which renders the functions $\sum_{i=0}^{m} \hat{a}_i f^i$, $a_i \in A$, continuous on $Z(\alpha(x), A)$, where $f(h, \lambda) \equiv \lambda$ for $(h, \lambda) \in Z(\alpha(x), A)$. From its definition, it is easily seen that π is continuous with respect to the prescribed topologies and it follows from Proposition 1.1 that π is also an open mapping. $\pi(Z(\alpha(x), A))$ need not be all of Φ_A; however, it is an open subset of Φ_A.

DEFINITION. A neighborhood W in $Z(\alpha(x), A)$ of (h_0, λ_0) is called a M-neighborhood of (h_0, λ_0) if, for each $h \in \pi(W)$,

$$M(h_0, \lambda_0) = \sum_{(h,\lambda) \in \pi^{-1}(h) \cap W} M(h, \lambda),$$

that is, $M(h_0, \lambda_0)$ is equal to the sum of the values of M at the points in $\pi^{-1}(h) \cap W$.

It follows from the above definition that if W is a M-neighborhood of (h, λ) and if V is a neighborhood in Φ_A of h, then $\pi^{-1}(V) \cap W$ is also a M-neighborhood of (h, λ).

PROPOSITION 1.1. *Let $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i \in A[x]$ and $(h_0, \lambda_0) \in Z(\alpha(x), A)$. If W is a neighborhood in $Z(\alpha(x), A)$ of (h_0, λ_0), then there exists a M-neighborhood W_0 of (h_0, λ_0) which is contained in W.*

Proof. It suffices to prove the proposition for the case where W is of the form $(V \times \{\lambda \in C : |\lambda - \lambda_0| < \varepsilon\}) \cap Z(\alpha(x), A)$, where $\varepsilon > 0$ and V is a neighborhood in Φ_A of h_0. Consider the polynomial $p(z_0, z_1, \cdots, z_n, w) = z_0 + z_1 w + \cdots + z_n w^n$. Then λ_0 is a root of $p(\hat{\alpha}_0(h_0), \hat{\alpha}_1(h_0), \cdots, \hat{\alpha}_n(h_0), w) = 0$. Suppose that λ_0 is a root of multiplicity m. From the Weierstrass preparation theorem, it follows that there exists $s > 0$ and $t > 0$ such that if $|z_i - \hat{\alpha}_i(h_0)| < t$, $i = 0, 1, \cdots, n$, then $p(z_0, z_1, \cdots, z_n, w) = 0$ possesses exactly m roots (multiplicities counted) in $\{w \in C : |w - \lambda_0| < s\}$ (cf. [5]). Furthermore, s can be so chosen that it is less than ε. Now, $V_0 = \{h \in \Phi_A : |\hat{\alpha}_i(h) - \hat{\alpha}_i(h_0)| < t,\ i = 0, 1, \cdots, n\}$ is a neighborhood in Φ_A of h_0 so that $V \cap V_0$ is also a neighborhood in Φ_A of h_0. Let W_0

denote the neighborhood $((V \cap V_0) \times \{\lambda \in C : |\lambda - \lambda_0| < s\}) \cap Z(\alpha(x), A)$. Then $W_0 \subset W$ and W_0 is a M-neighborhood of (h_0, λ_0). Q.E.D.

The above proposition tells us that the M-neighborhoods of a point in $Z(\alpha(x), A)$ form a base for the neighborhood system of that point.

From the construction in the proof of the proposition we have the following important result:

1.2. π is an open mapping.

This can be seen in the following way. Let W be an open subset of $Z(\alpha(x), A)$ and let $(h, \lambda) \in W$. Then let W_0 be the M-neighborhood of (h, λ) contained in W which was constructed above. It has the property that $\pi(W_0)$ is a neighborhood in Φ_A of h. Thus it follows that $\pi(W)$ is open and that π must be an open mapping.

Since the topology in $Z(\alpha(x), A)$ is Hausdorff, we can conclude from the proposition that any two distinct points of $Z(\alpha(x), A)$ possess disjoint M-neighborhoods. If the points have the same first coordinate, then M-neighborhoods can be found which have the same image under π. For if W_1 and W_2 are the disjoint M-neighborhoods of these points, then $V = \pi(W_1) \cap \pi(W_2)$ is a neighborhood so that $\pi^{-1}(V) \cap W_1$ and $\pi^{-1}(V) \cap W_2$ are the desired M-neighborhoods.

In the next section we concern ourselves with subsets K of $Z(\alpha(x), A)$ which are compact and open, and such that $\pi(K) = \Phi_A$. The next proposition shows such sets behave, with respect to Φ_A, in a manner which is reminiscent of covering spaces (cf. [3]). If $h \in \Phi_A$, let $r(h)$ denote the cardinality of $\pi^{-1}(h) \cap K$ and let $(h, \lambda_1), (h, \lambda_2), \cdots, (h, \lambda_{r(h)})$ denote the distinct points in $\pi^{-1}(h) \cap K$.

Proposition 1.3. *Let $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i \in A[x]$ and K be a compact open subset of $Z(\alpha(x), A)$ such that $\pi(K) = \Phi_A$. Then for each $h \in \Phi_A$, there are disjoint M-neighborhoods $V_k(h)$ of the points (h, λ_k) and positive numbers (depending on h) t and s_k, $k = 1, 2, \cdots, r(h)$ such that*

(i) for each k, $V_k(h)$ is contained in $\{(h', \lambda') \in K : |\hat{\alpha}_j(h') - \hat{\alpha}_j(h)| < t$, $j = 0, 1, \cdots, n$ and $|\lambda - \lambda_k| < s_k\}$,

(ii) for each k, $\pi(V_k(h)) = \pi(V(h_1))$, and

(iii) $\pi^{-1}(\pi(V_1)) \cap K = \bigcup_{k=1}^{r(h)} V_k(h)$.

Furthermore, if $U(h) = \{(z_0, z_1, \cdots, z_n) \in C^{n+1} : |z_j - \hat{\alpha}_j(h)| < t, j = 0, 1, \cdots, n\}$, then

(iv) $p(z_0, z_1, \cdots, z_n, w) = \sum_{i=0} z_i w^i$ has exactly $M(h, \lambda_k)$ roots (each distinct root repeated according to its multiplicity) in $\{\lambda \in C : |\lambda - \lambda_k| < s_k\}$ if $(z_0, z_1, \cdots, z_n) \in U(h)$.

Proof. Let $W_1, \cdots, W_{r(h)}$ be disjoint neighborhoods of the points $(h, \lambda_1), \cdots, (h, \lambda_{r(h)})$. In view of Proposition 1.1 (and its proof) and the fact that K is open, we may and do assume that the W_k's are M-neighborhoods satisfying (i) for an appropriate choice of $t > 0$ and $s_k > 0$, $k = 1, 2, \cdots, r(h)$, and such that (iv) is satisfied for this choice. As seen earlier, the openness of π allows us to assume that

$\pi(W_k) = \pi(W_1)$, $k = 1,2,\cdots,r(h)$. Now let $W = \pi^{-1}(\pi(W_1)) \cap K \sim \bigcup_{k=1}^{r(h)} W_k$ and consider the closure $\bar{W}$ of W. Since K is compact, $\bar{W}$ is compact and $\pi(\bar{W})$ is a compact subset of Φ_A; hence, $\pi(\bar{W})$ is a closed subset of Φ_A. Since $(h,\lambda_k) \notin \bar{W}$, for each k, $h \notin \pi(\bar{W})$ so that there is a neighborhood V in Φ_A of h such that $\pi(W) \cap V = \varnothing$; thus, $\pi^{-1}(V) \cap W = \varnothing$. If we take $V_k = W_k \cap \pi^{-1}(V)$, then the above properties are satisfied with t and the s_k's chosen as above. Q.E.D.

By M locally constant (π a local homeomorphism) at $(h,\lambda) \in Z(\alpha(x),A)$, we mean that there exists a neighborhood W in $Z(\alpha(x),A)$ of (h,λ) on which M is constant ($\pi | W : W \to \pi(W)$ is a homeomorphism).

LEMMA 1.4. *Let $\alpha(x) \in A[x]$. The following are equivalent:*

(i) *M is locally constant at $(h,\lambda) \in Z(\alpha(x),A)$,*

(ii) *π is a local homeomorphism at (h,λ), and*

(iii) *there exists a neighborhood V in Φ_A of h and a function $f \in C(V)$ such that $\alpha_{h'}(f(h')) = 0$ for each $h' \in V$, $M(\cdot,f(\cdot))$ is constant on V and $f(h) = \lambda$.*

Proof. We show (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii). Let W be a M-neighborhood of (h,λ) and let V_0 be a neighborhood in Φ_A of h such that $V_0 \subset V$ and $h' \in V_0$ implies that $(h',f(h')) \in W$. Then $\pi^{-1}(V_0) \cap W$ is a M-neighborhood of (h, λ) and hence we assume that $W = W \cap \pi^{-1}(V_0)$. Now, if $\pi | W$ is not a homeomorphism, it must be the case that there is a point $(h',\lambda') \in W$ such that $\lambda' \neq f(h')$ since π is an open mapping. But this means that $M(h',f(h')) < M(h',f(h')) + M(h',\lambda') \leqq M(h,f(h))$. This is a contradiction since $M(\cdot,f(\cdot))$ is constant on V and so it follows that $\pi | W$ must be a homeomorphism.

(ii) $\Rightarrow$ (i). Suppose that W is a neighborhood in $Z(\alpha(x),A)$ of (h,λ) such that $\pi | W$ is a homeomorphism. It may be assumed that W is a M-neighborhood since W contains such a neighborhood. Now, M must be constant on W. For if it is not, then there are two points (h',λ') and (h',λ'') in W, $\lambda' \neq \lambda''$. But this is a contradiction so that M is constant on W.

(i) $\Rightarrow$ (iii). Let W be a neighborhood in $Z(\alpha(x),A)$ of (h,λ) on which M is constant. We assume that W is a M-neighborhood of (h,λ). Thus, the cardinality of $\pi^{-1}(h') \cap W$ is always equal to one if $h' \in \pi(W)$. This means that $\pi | W$ is a homeomorphism so that if f is defined to be the function on $V = \pi(W)$ such that $(h',f(h')) \in W$, $h' \in V$, then f is continuous on V. Furthermore, $\alpha_{h'}(f(h')) = 0$, $h' \in V$, $f(h) = \lambda$ and $M(\cdot,f(\cdot))$ is constant on V. Q.E.D.

If K is a compact open subset of $Z(\alpha(x),A)$ such that $\pi(K) = \Phi_A$, then we define $m_K(h)$ to be the sum of the values of M at the points of $\pi^{-1}(h) \cap K$. (If we are dealing in any given situation with only one such set K, we will simply denote m_K by m.)

LEMMA 1.5. *If K is a compact open subset of $Z(\alpha(x),A)$ such that $\pi(K) = \Phi_A$, then there exists disjoint open subsets X_i, $i = 1, 2, \cdots, s$, of Φ_A such that $\bigcup_{i=1}^{s} X_i = \Phi_A$ and the function $m(\cdot)$ is constant on each X_i.*

Proof. For each $h \in \Phi_A$, let $(h,\lambda_1(h)),\cdots,(h,\lambda_{r(h)}(h))$ denote the points in $\pi^{-1}(h) \cap K$ and let $X_i = \{h \in \Phi_A : m(h) = i\}$. Then the X_i are open sets in Φ_A. For if $h \in X_i$, for some i, then let V_k, $k = 1,2,\cdots,r(h)$, be the neighborhoods of the (h,λ_k)'s which were given in Proposition 1.3. Then $h' \in \pi(V_1)$ implies that

$$m(h') = \sum_{i=1}^{r(h')} M(h',\lambda(h')) = \sum_{i=1}^{r(h)} \left(\sum_{(h',\lambda') \in \pi^{-1}(h') \cap V_i} M(h',\lambda') \right)$$

$$= \sum_{i=1}^{r(h)} M(h,\lambda_i(h)) = m(h)$$

so that $h' \in X_i$ and hence $\pi(V_1) \subset X_i$. Thus, each X_i is open in Φ_A. Clearly, the X_i's are disjoint. Q.E.D.

We now turn our attention to the case where $\alpha(x)$ is a monic polynomial. In this case, $Z(\alpha(x),A)$ is compact and its image under π is all of Φ_A. ($Z(\alpha(x),A)$ is, for monic polynomials, the carrier space of the Banach algebra $A[x]/(\alpha(x))$; see [2] for details. If M is locally constant on $Z(\alpha(x),A)$, then $Z(\alpha(x),A)$ and Φ_A must both decompose topologically in the way specified in

Lemma 1.6. *Suppose that $\alpha(x) \in A[x]$ is a monic polynomial. If M is locally constant on $Z(\alpha(x),A)$, then there exist disjoint open sets X_i, $i = 1,2,\cdots,s$, in Φ_A and disjoint open sets Y_{ij} in $Z(\alpha(x),A)$, $j = 1,2,\cdots,s_i$; $i = 1,2,\cdots,s$, such that* (i) $\Phi_A = \bigcup_{i=1}^{s} X_i$, (ii) $\pi^{-1}(X_i) = \bigcup_{j=1}^{s_i} Y_{ij}$, (iii) card $\pi^{-1}(h) \cap Y_{ij}$ *is constant on X_i for each i and j and* (iv) *M is constant on each Y_{ij}.*

Proof. Let F_j denote the set $\{(h,\lambda) \in Z(\alpha(x),A) : M(h,\lambda) = j\}$. Since M is locally constant on $Z(\alpha(x),A)$, the F_j's are open subsets of $Z(\alpha(x),A)$ and hence closed since they are disjoint. Now, let G_{ij} denote the set $\{h \in \Phi_A : \text{card}(\pi^{-1}(h) \cap F_j) = i\}$. We will show that G_{ij} is both open and closed by showing that if h is in the closure $\bar{G}_{ij}$ of G_{ij}, then h is in the interior of G_{ij}. To this end, let $h \in \Phi_A$ and let (h,λ_k), $k = 1,2,\cdots,r(h)$, denote the points in $\pi^{-1}(h)$. We can find disjoint neighborhoods V_k in $Z(\alpha(x),A)$ of the points (h,λ_k) such that for each k (i) $\pi | V_k : V_k \to \pi(V_k)$ is a homeomorphism, (ii) M is constant on V_k and (iii) $\pi(V_k) = \pi(V_1)$. Furthermore, (iv) $\pi^{-1}(\pi(V_1)) = \bigcup_{k=1}^{r(h)} V_k$. To see this, take M-neighborhoods W_k of the (h,λ_k) which satisfy conditions (ii) and (iv) of Proposition 1.3. In view of Lemma 1.4, we know there exists, for each k, a neighborhood U_k in $Z(\alpha(x),A)$ of (h,λ_k) such that M is constant on U_k and $\pi | U_k : U_k \to \pi(U_k)$ is a homeomorphism. Let $V = \bigcap_{k=1}^{r(h)} \pi(U_k)$. If V_k is taken to be $\pi^{-1}(V) \cap W_k$, then the above conditions (i)–(iv) are satisfied.

Now, if $h \in \bar{G}_{ij}$, then there exists a $h' \in \pi(V_1) \cap G_{ij}$ and hence there are distinct points $(h',\lambda_1'),\cdots,(h',\lambda_i')$ in F_j. Each point (h',λ_t), $1 \leqq t \leqq i$, belongs to one and only one of the V_k's. Denote the subscript by $k(t)$. Then $M \equiv j$ on $V_{k(t)}$. Furthermore, if $M \not\equiv j$ on V_k, then $V_k \cap V_{k(t)} = \varnothing$, $t = 1,2,\cdots,i$. Thus, card$(\pi^{-1}(h) \cap F_j) = i$ and h must be in G_{ij}. Also, by the same argument, $\pi(V_1) \subset G_{ij}$ so that G_{ij} is both open and closed.

For each $h \in \Phi_A$, let $V(h) = \bigcap_{h \in G_{ij}} G_{ij}$. Since there are only finitely many G_{ij}'s the $V(h)$'s are both open and closed in Φ_A. We next show that if $h' \in V(h)$, then $V(h') = V(h)$. For clearly, $V(h') \subset V(h)$. Let $G_{i_1,j_1}, \cdots, G_{i_n,j_n}$ be the set of all G_{ij}'s whose intersection is $V(h)$. Then $\sum_{k=1}^{n} i_k j_k = m(h)$ ($m(\cdot)$ defined above). If the intersection $V(h')$ is smaller than $V(h)$, then

$$m(h) = \sum_{k=1}^{n} i_k j_k < \sum_{(h',\lambda') \in \pi^{-1}(h')} M(h',\lambda').$$

But the right-hand side is equal to $m(h')$. Observing that $m(\cdot)$ is constant on Φ_A, we have a contradiction. Thus, $V(h) = V(h')$.

Since the sets $V(h)$ are open and either identical or disjoint, the compactness of Φ_A implies that there are points $h_1, h_2, \cdots, h_s$ in Φ_A such that $\Phi_A = \bigcup_{i=1}^{s} V(h_i)$ and $V(h_i) \cap V(h_j) = \varnothing$ if $i \neq j$. Let $X_i = V(h_i)$ and $Y_{ij} = \pi^{-1}(X_i) \cap F_j$. With this choice, conditions (i)–(iv) of the lemma are satisfied. Q.E.D.

2. **Factorization of polynomials.** Let us recall that we are assuming that our Banach algebras are semi-simple. Examples can be given which show that this assumption is a necessary one.

Suppose that $\alpha(x)$, $\alpha_i(x) \in A[x]$, $i = 1,2$, and that $\alpha(x) = \alpha_1(x)\alpha_2(x)$. Then $Z(\alpha(x),A) = Z(\alpha_1(x),A) \cup Z(\alpha_2(x),A)$. (We are assuming that not all the coefficients of $\hat{\alpha}(x)$ vanish at the same points of Φ_A.) Associated with each set $Z(\alpha_i(x),A)$ we have the multiplicity function M_i of the polynomial $\alpha_i(x)$, and if we agree that M_i is to be zero at those points of $Z(\alpha(x),A)$ for which it is not defined, then $M_1(h,\lambda) + M_2(h,\lambda) = M(h,\lambda)$, $(h,\lambda) \in Z(\alpha(x),A)$. Furthermore, the sets $Z(\alpha_i(x),A)$ are closed in the prescribed topology in $Z(\alpha(x),A)$.

Suppose now that there exist closed subsets X_1 and X_2 of $Z(\alpha(x),A)$ such that $Z(\alpha(x),A) = X_1 \cup X_2$ and functions M_1 and M_2 which behave like multiplicity functions on X_1 and X_2, respectively, and $M_1 + M_2 = M$. Thus, we ask: is this enough to force a factorization over A of $\alpha(x)$, with factors $\alpha_1(x)$ and $\alpha_2(x)$ such that $Z(\alpha_i(x),A) = X_i$, $i = 1,2$. The answer is yes if $A = C(\Omega)$ but is no in general. In fact, if A is regular and self-adjoint the above conditions are still not sufficient. Witness the following example. Let $A = L_1(I)$, $I =$ integers, let $f \in A$ such that $|\hat{f}| \notin \hat{A}$, and let $\alpha(x) = x^2 - g$, $\hat{g} = |\hat{f}|^2$. Then the above conditions are satisfied if $X_i = \{(h,\lambda) \in \Phi_A \times C : \lambda = (-1)^i |\hat{f}(h)|\}$ and $M_i \equiv 1$, $i = 1,2$, but $\alpha(x)$ fails to factor into a product $\alpha_1(x)\alpha_2(x)$, $Z(\alpha_i(x),A) = X_i$, $i = 1,2$, since this would imply that $|\hat{f}| \in \hat{A}$. Thus, we see that the factorization of a polynomial $\alpha(x)$ over A is not, in general, a topological property of the space $Z(\alpha(x),A)$. However, if $X_1 \cap X_2 = \varnothing$, then $\alpha(x)$ must factor as indicated above. Before proving this (Theorem 2.1) we set the following notation.

Let $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i \in A[x]$. We will use (z) for $(z_0, z_1, \cdots, z_n)$, $((z),\lambda)$ for $(z_0, z_1, \cdots, z_n, \lambda)$ and $(z(h))$ for $(\hat{\alpha}_0(h), \hat{\alpha}_1(h), \cdots, \hat{\alpha}_n(h))$, $h \in \Phi_A$. If K is a compact open subset of $Z(\alpha(x),A)$ such that $\pi(K) = \Phi_A$, then for each $h \in \Phi_A$, let

$\pi^{-1}(h) \cap K = \{(h,\lambda_1(h)),\cdots,(h,\lambda_{r(h)}(h))\}$. Let $V_i(h) \subset K$ denote the M-neighborhood of $(h,\lambda_i(h))$ and $U(h)$ the neighborhood in C^{n+1} of $(z(h))$ which were specified in Proposition 1.3. Furthermore, let $U_i(h)$ denote the set

$$\{((z),\lambda) \in U(h) \times C : p(z_0,z_1,\cdots,z_n,\lambda) = \sum_{i=0}^{n} z_i\lambda^i = 0, \text{ and } |\lambda - \lambda_i| < s_i\}$$

(see Proposition 1.3 for the definition of s_i; s_i depends on h), i varying from one to $r(h)$.

THEOREM 2.1. *Let $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i \in A[x]$. If K is a compact open subset of $Z(\alpha(x),A)$ such that $\pi(K) = \Phi_A$, then there are mutually orthogonal idempotents e_i in A and polynomials $\beta_i(x)$, $Q_i(x) \in A[x]$, $i = 1, 2, \cdots, s$, such that* (i) $e = \sum_{i=1}^{s} e_i$, (ii) $e_i\beta_i(x)$ *is monic over* e_iA, (iii) $K = \bigcup_{i=1}^{s} Z(e_i\beta_i(x), e_iA)$, *and* (iv) $\alpha(x) = (\sum_{i=1}^{s} e_i\beta_i(x))(\sum_{i=1}^{s} e_iQ_i(x))$.

Proof. For simplicity (of notation), let us first assume that the function $m(\cdot)$ (defined in the last section) is constant on Φ_A. We use m itself to denote this constant.

For each $h \in \Phi_A$, let $f_j(h)$ be defined by

$$f_j(h) = (-1)^{m-j}(\lambda_1(h)\lambda_2(h)\cdots\lambda_{m-j}(h) + \lambda_1(h)\lambda_2(h)\cdots\lambda_{m-j-1}(h)\lambda_{m-j+1}(h) + \cdots)$$

for $j = 0,1,\cdots,m-1$. Clearly $f_j \in C(\Phi_A)$ for each j. The next step is to show that $f_j \in \hat{A}$.

For each $h \in \Phi_A$, let $F_{j,h}$ denote the function defined by

$$F_{j,h}(z) = (-1)^{m-j}(\mu_1(z)\mu_2(z)\cdots\mu_{m-j}(z) + \mu_1(z)\mu_2(z)\cdots\mu_{m-j-1}(z)\mu_{m-j+1}(z) + \cdots)$$

for $j = 0,1,\cdots,m-1$, where $(z) \in U(h)$ and $\mu_k(z)$ denotes the roots of $p(z_0,z_1,\cdots,z_n,\mu) = 0$, each repeated according to its multiplicity, such that $((z),\mu_k(z)) \in \bigcup_{i=1}^{r(h)} U_i(h)$. Each $F_{j,h}$ is analytic in $U(h)$ (cf., for example, [4]). Furthermore, if $h' \in V(h)$, then $F_{j,h} = F_{j,h'}$ in $U(h) \cap U(h')$, and $F_{j,h}(z(h)) = f_j(h)$ for each $h \in \Phi_A$. (This says that $h \to F_{j,h}$ is a continuous mapping in the sense described in [1].) Hence by a theorem (6.2, [1]) of Arens and Calderón, $f_j \in \hat{A}$.

Now let $\beta_i \in A$ be the unique elements such that $\hat{\beta}_j = f_j$ for each j. If $\beta(x)$ denotes the monic polynomial $\sum_{i=0}^{m-1} \beta_i x^i + x^m$, then $Z(\beta(x),A) = K$ and the multiplicity function of $\beta(x)$ is precisely $M|K$, where M is the multiplicity function of $\alpha(x)$. Since $\beta(x)$ is monic, there are polynomials $Q(x)$, $R(x) \in A[x]$ such that $\alpha(x) = \beta(x)Q(x) + R(x)$ and the degree of $R(x)$ is less than that of $\beta(x)$. Now, for each $h \in \Phi_A$, we have that $\alpha_h(x) = \beta_h(x)Q_h(x) + R_h(x)$ so that $R_h(x) = 0$ since $\beta_h(x) = 0$ has m roots (including multiplicities). Hence $R(x) = 0$ and $\alpha(x) = \beta(x)$ $Q(x)$.

We now turn to the more general situation of the theorem where m is not necessarily constant on Φ_A. However, in this case m is locally constant on Φ_A

(cf. Lemma 1.5). Let X_i, $i = 1,2,\cdots,s$, be the (disjoint) sets of constancy of the function m. By a theorem of Šilov (cf. [6]), there exist mutually perpendicular idempotents $e_1, e_2, \cdots, e_s$, such that $e = \sum_{i=1}^{s} e_i$ and $\hat{e}_i(h) = 1$ if and only if $h \in X_i$. Also, Φ_{e_iA} is (identifiable with) X_i. Applying the first part of the proof to $e_i\alpha(x)$ over e_iA, for each i, we have polynomials $\beta_i(x)$ and $Q_i(x)$ over A such that $e_i\alpha(x) = e_i\beta_i(x)Q_i(x)$, $e_i\beta_i(x)$ monic over e_iA. Thus $(\sum_{i=1}^{s} e_i\beta_i(x))(\sum_{i=1}^{s} e_iQ_i(x)) = \sum_{i=1}^{s} e_i\alpha(x) = \alpha(x)$. It is clear that $Z(\sum_{i=1}^{s} e_i\beta_i(x), A) = K$. Q.E.D.

The factorization demonstrated above is not unique. For example, when $s = 2$,

$$\alpha(x) = (e_1\beta_1(x) + e_2\beta_2(x))(e_1Q_1(x) + e_2Q_2(x))$$
$$= (e_1\beta_1(x) + e_2Q_2(x))(e_1Q_1(x) + e_2\beta_2(x)).$$

We will take up this question of uniqueness in Theorem 2.4.

We now turn to the question of when a factor appears as a multiple factor If $\alpha(x), \beta(x) \in A[x]$ and if k is a positive integer such that $\beta(x)^k = \alpha\ x)$, then $M(h,\lambda)/k$ is an integer for every $(h,\lambda) \in Z(\beta(x),A)$ ($Z(\beta(x),A)$ is a subset of $Z(\alpha(x),A)$). Without any assumptions on $\alpha(x)$, the converse is false, that is, the existence of such a k need not imply that there exists a polynomial $\beta(x)$ such that $\beta(x)^k = \alpha(x)$. Witness the following example. Let $\Delta = \{\lambda \in C : |\lambda| \leqq 1\}$ and $A = \{f \in C(\Delta) : f$ analytic in the interior of Δ and $f'(0) = 0\}$. If $\alpha(x) = z^2x^2 - 2z^3x + z^4$, then M is identically equal to two on $Z(\alpha(x),A)$ but $\alpha(x)$ is not the square of any other polynomial. This difficulty vanishes if we assume that $\alpha(x)$ is monic over A. That this is true follows from the corollary below, which is stated in a more general setting for later use. In the remainder of this section, we restrict our attention to monic polynomials.

In the following corollary, $Z(\alpha(x),A)$ is assumed to be the union of disjoint compact open subsets X_i, $i = 1,2,\cdots,t$. The functions $m_{X_i}(\cdot)$ defined earlier will be denoted by $m_i(\cdot)$·

Corollary 2.2. *Let $\alpha(x)$ be a monic polynomial over A and $X_1, X_2, \cdots, X_t$, be disjoint open subsets of $Z(\alpha(x),A)$, the union of which is $Z(\alpha(x),A)$. Assume that each $m_i(\cdot)$ is constant on Φ_A and denote its constant value by m_i. If there are positive integers $k_1, k_2, \cdots, k_t$, such that for each i and every $(h,\lambda) \in X_i$, $M(h,\lambda)/k_i$ is an integer, then there are monic polynomials $\alpha_i(x)$, $i = 1,2,\cdots,t$, of degree m_i/k_i over A such that $Z(\alpha_i(x),A) = X_i$ and $\alpha(x) = \prod_{i=1}^{t} \alpha_i(x)^{k_i}$.*

Proof. By repeated applications of the theorem,there are monic polynomials $\beta_1(x), \beta_2(x), \cdots, \beta_t(x) \in A[x]$ such that $\alpha(x) = \prod_{i=1}^{t} \beta_i(x)$ and $Z(\beta_i(x),A) = X_i$. The degree of $\beta_i(x)$ is m_i and the multiplicity function of $\beta_i(x)$ is $M_i = M | X_i$. Recall that the coefficient β_{ij} of X_j in $\beta_i(x)$ satisfies

$$\hat{\beta}_{ij}(h) = (-1)^{m_i - j}(\lambda_1^{(i)}(h) \cdots \lambda_{m_i - j}^{(i)}(h) + \cdots),$$

where $\lambda_1^{(i)}(h), \cdots, \lambda_{m_i}^{(i)}(h)$ are all the roots of $(\beta_i)_h(x) = 0$, each distinct root repeated according to its multiplicity. Let $\mu_1^{(i)}(h),\cdots,\mu_{n_i}^{(i)}(h)$, $n_i = m_i/k_i$, denote all the roots of $(\beta_i)_h(x) = 0$ but now each distinct $\mu_j^{(i)}(h)$ is to be repeated only $M(h,\mu_j^{(i)}(h))/k_i$ times. Then the functions $\alpha_{i,j}$ defined by

$$\alpha_{i,j}(h) = (-1)^{n_i - j}(\mu_1^{(i)}(h)\cdots\mu_{n_i-j}^{(i)}(h) + \cdots), \qquad j = 0,1,2,\cdots,n_i - 1,$$

are continuous on Φ_A. Furthermore, if $\alpha_i(x) = \sum_{j=0}^{n_i-1}\alpha_{i,j}x^i + x^{n_i}$, then $\alpha_i(x)^{k_i} = \sum_{i=0}^{m_i}\hat{\beta}_{ij}x^i$. But this means that the coefficients of $\alpha_i(x)$ must lie in $\hat{A}$; for the coefficient of x^{m_i-l} in $\alpha_i(x)^{k_i}$, $n_i = m_i/k_i \geqq l \geqq 1$, is of the form

$$k_i(\alpha_{i\ n_i-l}) + (\text{terms in } \alpha_{ij},\ n_i > j > n_i - l)$$

so that by induction the coefficients of $\alpha_i(x)$ must lie in $\hat{A}$. Hence, $\alpha(x) = \prod_{i=1}^{t}\alpha_i(x)^{k_i}$. Since A is semi-simple, we may and do take $\alpha_i(x)$ to be a polynomial over A. This proves the corollary since the $\alpha_i(x)$ are monic, of degree $n_i = m_i/k_i$ and $Z(\alpha_i(x),A) = X_i$.

We now turn our attention to the case where M is locally constant on $Z(\alpha(x),A)$, with $\alpha(x)$ monic.

THEOREM 2.3. *Let $\alpha(x)$ be a monic polynomial over A. If M is locally constant on $Z(\alpha(x),A)$, then there are mutually orthogonal idempotents $e_i \in A$, positive integers k_{ij} and polynomials $\alpha_{ij}(x) \in A[x]$, $j = 1,2,\cdots,s_i$; $i = 1,2,\cdots,s$, such that $e_i\alpha_{ij}(x)$ is monic over e_iA, the discriminant of $\prod_{j=1}^{s_i}e_i\alpha_{ij}(x)$ is inevertible in e_iA and*

$$\alpha(x) = \sum_{i=1}^{s} e_i \prod_{j=1}^{s_i}\alpha_{ij}(x)^{k_{ij}}.$$

Furthermore, if $\gamma(x)$ is any other polynomial over A such that $\gamma_h(\lambda) = 0$ for each $(h,\lambda) \in Z(\alpha(x),A)$, then $\gamma(x)$ is a multiple of $\beta(x) = \sum_{i=1}^{s} e_i \prod_{j=1}^{s_i}\alpha_{ij}(x)$.

Proof. Let X_i, Y_{ij}, $j = 1,2,\cdots,s_i$; $i = 1,2,\cdots,s$ be the sets constructed in Lemma 1.6. Let k_{ij} be equal to the constant value of M on Y_{ij}. It follows from Theorem 2.1 and its corollary that there are mutually orthogonal idempotents $e_i,\cdots,e_s$ and polynomials $\alpha_{ij}(x)$ over A such that $\sum_{i=1}^{s}e_i = e$,

$$e_i\alpha(x) = \prod_{j=1}^{s_i} e_i\alpha_{ij}(x)^{k_{ij}}$$

and $Y_{ij} = \{(h,\lambda) \in X_i \times C : (\alpha_{ij})_h(\lambda) = 0\}$, $i = 1,2,\cdots,s$. The degree of $e_i\alpha_{ij}(x)$ is equal to

$$\sum_{(h,\lambda)\in\pi^{-1}(h)\cap Y_{ij}} (M(h,\lambda)/k_{ij})$$

which in turn is equal to the number of points in $\pi^{-1}(h) \cap Y_{ij}$, $h \in X_i$. Hence there are as many distinct roots of $(\alpha_{ij})_h(x) = 0$, $h \in X_i$, as the degree of $e_i\alpha_{ij}(x)$

so that the discriminant of $e_i\alpha_{ij}(x)$ is invertible in e_iA. Since the Y_{ij} are disjoint, the discriminant of $e_i\prod_{j=1}^{s_i} \alpha_{ij}(x)$ is invertible in e_iA.

Now suppose that $\gamma_h(\lambda)=0$ for all $(h,\lambda)\in Z(\alpha(x), A)$. Then $e_i\gamma(x) = e_i\beta(x)Q_i(x) + e_iR_i(x)$ for some $Q_i(x)$ and $R_i(x)$ in $A[x]$, where the degree of $e_iR_i(x)$ is less than the degree of $e_i\beta(x)$ (defined above). Since $\beta_h(x) = 0$ has no multiple roots for $h \in X_i$, $(R_i)_h(x) = 0$ has a greater number of distinct roots than its degree so that the coefficients of $e_iR_i(x)$ belong to every maximal ideal of e_iA. Hence $R_i(x) = 0$ since A is semi-simple and

$$\gamma(x) = \left(\sum_{i=1}^{s} e_iQ_i(x)\right)\beta(x). \qquad \text{Q.E.D.}$$

The last assertion of the above theorem says that, in other words, if M is locally constant on $Z(\alpha(x), A)$, with $\alpha(x)$ monic, then the radical $A[x]/(\alpha(x))$ is a principal ideal generated by the coset $\beta(x) + (\alpha(x))$ (cf. [2]).

We return now to the question of uniqueness of the factorization. Our earlier comments showed that if A is the direct sum of two of its ideals e_1A and e_2A, $e_ie_j = \delta_{ij}e_i$, $e = e_1 + e_2$, then the factorization is not unique. If A is not decomposable, the factorization is still not necessarily unique. In this case M is not locally constant on $Z(\alpha(x),A)$, as following theorem shows.

Theorem 2.4. *Suppose that A is an indecomposable algebra. If $\alpha(x)$ is a monic polynomial over A and if its multiplicity function M is locally constant on $Z(\alpha(x),A)$, then a factorization of $\alpha(x)$ into irreducible factors is unique (to within units).*

Proof. Recall first that $Z(\alpha(x),A)$ is compact since $\alpha(x)$ is monic. In this situation it is well known that the mapping π from $Z(\alpha(x),A)$ onto Φ_A is a closed mapping.

Since A is indecomposable, we know that Φ_A is connected (cf. [7]). Hence $Z(\alpha(x),A)$ must be the union of disjoint open connected subsets X_i such that $\pi(X_i) = \Phi_A$. For if $Z(\alpha(x),A) = Q_1 \cup Q_2$, where Q_1 and Q_2 are disjoint and open, and hence closed, then $\pi(Q_i)$ is both open and closed in Φ_A so that $\pi(Q_i) = \Phi_A$ or $Q_i = \varnothing$. This process of decomposition, when repeated, must terminate after a finite number of steps so that there are disjoint open connected sets $X_1, X_2, \cdots, X_t$ in $Z(\alpha(x),A)$ such that $\bigcup_{i=1}^{t} X_i = \Phi_A$ and $\pi(X_i) = \Phi_A$ for each i. Furthermore, the assumption that M is locally constant on $Z(\alpha(x),A)$ implies that M is constant on each X_i. This assumption together with the fact that X_i is connected implies that the number of points in $\pi^{-1}(h) \cap X_i$, for each i, is constant on Φ_A. Let k_i denote the constant value of M on X_i and n_i the card$(\pi^{-1}(h) \cap X_i)$. Applying Theorem 2.3, we have that there are monic polynomials $\alpha_i(x) \in A[x]$ such that the degree of $\alpha_i(x)$ is n_i, $\alpha(x) = \prod_{i=1}^{t} \alpha_i(x)^{k_i}$ and $Z(\alpha_i(x),A) = X_i$. Also, the discriminant of each $\alpha_i(x)$ is invertible in A. That the factors $\alpha_i(x)$ are irreducible follows from the argument given below.

Suppose that $\alpha(x) = \beta_1(x)\beta_2(x) \cdots \beta_n(x)$. We can assume without loss of generality that $n=2$. Then $Z(\alpha(x),A)=Z(\beta_1(x),A)\cup Z(\beta_2(x),A)$ and the $Z(\beta_i(x),A)$ are closed in $Z(\alpha(x),A)$. We now show that each $Z(\beta_i(x),A)$ is also open. Let M_i denote the function on $Z(\alpha(x),A)$ which is equal to the multiplicity function of $\beta_i(x)$ on $Z(\beta_i(x),A)$ and zero otherwise. Then $M_1(h,\lambda)+M_2(h,\lambda)\equiv M(h,\lambda)$. Each point $(h,\lambda)\in Z(\beta_1(x),A)$ possesses a neighborhood W_0 in $Z(\alpha(x),A)$ on which M_1 is constant. To see this, choose a neighborhood W_0 in $Z(\alpha(x),A)$ of (h,λ), on which M is constant, so that $W_0\cap Z(\alpha(x),A)$ is a M-neighborhood (relative to the polynomial $\beta_1(x)$) in $Z(\beta_1(x),A)$ of (h,λ) and $W_0\cap Z(\beta_2(x),A)$ (relative to the polynomial $\beta_2(x)$) is a M-neighborhood in $Z(\beta_2(x),A)$ of (h,λ) if $(h,\lambda)\in Z(\beta_2(x),A)$ but $W_0\cap Z(\beta_2(x),A)=\varnothing$ if $(h,\lambda)\notin Z(\beta_2(x),A)$. Then $M_i(h',\lambda)\leqq M_i(h,\lambda)$ for each $(h',\lambda')\in W_0$ and $i=1,2$. If M_1 is not constant on W_0, then there is a point $(h',\lambda')\in W_0$ such that $M_1(h',\lambda')<M_1(h,\lambda)$. By the above restrictions on W_0, $M_2(h',\lambda')\leqq M_2(h,\lambda)$ so that $M(h',\lambda')=M_1(h',\lambda')+M_2(h',\lambda')$ $<M_1(h,\lambda)+M_2(h,\lambda)=M(h,\lambda)$. This is a contradiction and so it must be the case that M_i is constant on W_0. Since $M_1(h,\lambda)>0$, W_0 is a subset of $Z(\beta_1(x),A)$. Thus, $Z(\beta_1(x), A)$ is open, as well as closed, in $Z(\alpha(x), A)$. This means that $Z(\beta_1(x),A)$ is the union of some of the X_i, say $X_1,X_2,\cdots,X_s$ (after possible relabeling) and fails to intersect the other X_i's. $\prod_{i=1}^{s}\alpha_i(x)$ is a monic polynomial with an invertible discriminant since the $Z(\alpha_i(x),A)$'s are disjoint. It follows from the previous theorem that $\prod_{i=1}^{s}\alpha_i(x)$ is a factor of $\beta_1(x)$.

By continuing this process we have that

$$\beta_j(x)=Q_j\prod_{i=1}^{t}\alpha_i(x)^{k_{ij}},\qquad j=1,2,$$

for an appropriate choice of non-negative integers k_{ij} and $Q_j\in A$. Since $Q_1Q_2=e$, the Q_i are invertible in A. Q.E.D.

3. **Zeros of polynomials.** In this section we deal with sufficient conditions under which $\alpha(x)\in A[x]$ has a zero in (i) a Banach algebra extension of A, that is, a Banach algebra B in which A is embedded isomorphically and isometrically, and (ii) in A itself. In [2] it is shown that if A is a Banach algebra and if $\beta(x)\in A[x]$ has an invertible leading coefficient, then $A[x]/(\beta(x))$ possesses a norm which makes it a Banach algebra extension of A, the isomorphism being the canonical one.

It is also shown in [2] that if A is semi-simple and if R denotes the radical of $A[x]/(\beta(x))$, then $(A[x]/(\beta(x)))/R$ possessess a norm so that it is a Banach algebra extension of A. Again the isomorphism is the canonical one. We now make use of these propositions in proving

THEOREM 3.1. *Let A be a semi-simple Banach algebra and let $\alpha(x)$ be a polynomial over A. If K is a compact open subset of $Z(\alpha(x), A)$ such that $\pi(K)=\Phi_A$, then there exists a Banach algebra extension $A[b]$ of A such that $\alpha(b)=0$.*

Proof. By Theorem 2.1, $\alpha(x) = (\sum_{i=1}^{s} e_i\alpha_i(x))(\sum_{i=1}^{s} e_iQ_i(x))$ where the e_i are mutually orthogonal idempotents such that $e = \sum_{i=1}^{s} e_i$. Furthermore, the $e_i\alpha_i(x)$ are monic over e_iA and $Z(\sum_{i=1}^{s} e_i\alpha_i(x), A) = K$. Let m denote the least common multiple of the n_i, n_i = degree of $e_i\alpha_i(x)$, and let $\beta(x) = \sum_{i=1}^{s} e_i\alpha_i(x)^{m/n_i}$. Then $\beta(x)$ is a monic polynomial over A and hence has a solution in $A[x]/(\beta(x))$. Now $(A[x]/(\beta(x)))/R$ is isomorphic to $(A[x]/(\beta(x)))^\wedge$, the Gelfand representation of $A[x]/(\beta(x))$. Thus, $(A[x]/(\beta(x)))^\wedge$ is of the form $\hat{A}[b]$, where $b = (x + (\beta(x)))^\wedge$, and $\sum_{i=1}^{s} \hat{e}_i\hat{\alpha}_i(b) = 0$. By the above comments, $(A[x]/(\beta(x)))/R$ is a Banach algebra extension of A; hence, $\hat{A}[b]$ is a Banach algebra extension of A. Q.E.D.

Our next theorem gives a sufficient condition that $\alpha(x) = 0$ has a solution in A, where $\alpha(x) \in A[x]$, and generalizes a theorem (in case A is semi-simple, which we assume here) due to Arens and Calderón.

THEOREM 3.2. *Let A be a semi-simple Banach algebra. If $\alpha(x) \in A[x]$, if $f \in C(\Phi_A)$ such that $\hat{\alpha}(f) = 0$ and if $M(\cdot, f(\cdot))$ is locally constant on Φ_A, then $f \in \hat{A}$, that is, there is an $a \in A$ such that $\alpha(a) = 0$ and $\hat{a} = f$.*

Proof. Let K denote the graph of f. Then K is a compact subset of $Z(\alpha(x), A)$. By Lemma 1.4, K is an open subset of $Z(\alpha(x), A)$ so that by Theorem 2.1 and its corollary,

$$\alpha(x) = \sum_{i=1}^{s} e_i\alpha_i(x)^{k_i}Q_i(x)$$

where $K = Z(\sum_{i=1}^{s} e_i\alpha_i(x), A)$ and the degree of $e_i\alpha_i(x)$ is one. If $e_i\alpha_i(x) = e_ix - e_ib_i$, then $\sum_{i=1}^{s}(e_ib_i)^\wedge$ must be f. Thus $f \in \hat{A}$, or equivalently, there exists $a \in A$ such that $\alpha(a) = 0$ and $\hat{a} = f$. Q.E.D.

The Arens-Calderón theorem (4.1, [1]) assumes that $M(\cdot, f(\cdot))$ is identically one but does not require that A be semi-simple. Precisely: if $\sum_{i=0}^{n} \hat{a}_i(h)(f(h))^i = 0$ and $M(h, f(h)) = 1$ for each $h \in \Phi_A$, then there exists $a \in A$ such that $\sum_{i=0}^{n} \alpha_i a^i = 0$ and $\hat{a} = f$. When $M(\cdot, f(\cdot))$ is just locally constant on Φ_A and A not semi-simple, we can not draw the stronger conclusion that there is an $a \in A$ such that $\alpha(a) = 0$ and $\hat{a} = f$.

REMARK. Some of the main theorems of the Arens-Calderón paper [1] deal with functions $G(z_0, z_1, \cdots, z_n, w)$ which are analytic in a neighborhood of $K = \{(\hat{\alpha}_0(h), \hat{\alpha}_1(h), \cdots, \hat{\alpha}_n(h), f(h)) : h \in \Phi_A\}$. Specifically, they show that if $G(z_0, z_1, \cdots, z_n, w) = 0$, $G_w(z_0, z_1, \cdots, z_n, w) \neq 0$ for each $(z_0, z_1, \cdots, z_n, w) \in K$, then $f \in \hat{A}$ (cf. Theorem 6.8, [1]). The condition on G_w can be replaced by the condition that the multiplicity of $f(h)$ as a root of $G(\hat{\alpha}_0(h), \hat{\alpha}_1(h), \cdots, \hat{\alpha}_n(h), \lambda) = 0$ is locally constant on Φ_A. If so, then f once again must belong to $\hat{A}$. The proof of Theorem 3.2 can be extended so as to give a proof of the above assertion.

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